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Geometrization of the Gauge Connection within a Kaluza–Klein Theory

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We geometrize a generic (abelian and non-abelian) gauge coupling within the framework of a Kaluza–Klein theory, by choosing a suitable matter-field dependence on the extra coordinates. We first extend the Nöther theorem to a multidimensional spacetime, the Cartesian product of a 4-dimensional Minkowski space and a compact homogeneous manifold (whose isometries reflect the gauge symmetry). On such a "vacuum" configuration, the extra-dimensional components of the field momentum correspond to the gauge charges. Then we analyze the structure of a Dirac algebra for a spacetime with the Kaluza–Klein restrictions. By splitting the corresponding free-field Lagrangian, we show how the gauge coupling terms arise.

KEY WORDS: gauge theories; kaluza-klein theories.
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1. BASIC STATEMENTS

The works of Kaluza (1921) and Klein (1926a,b) allowed one to include the electromagnetic field within a geometrical picture, by adding an extra space-like dimension to the spacetime; in spite of this success, the Kaluza–Klein theories had their full development only after the formulation of non-abelian gauge theories (Mendl and Shaw, 1984; Coleman, 1988). In fact, the main achievement of this approach relies on the geometrization of Yang–Mills fields, whose group of symmetry admits a representation in terms of an isometry in the extra-dimensions (Cho and Freund, 1975; Cho, 1975). (For a complete discussion of this topic, see the works collected in (Appelquist *et al.*, 1987) or the review presented in (Overduin and Wesson, 1997); see also (Cremmer *et al.*, 1978) for an extension of the multidimensionality idea to supergravity theory.)

The price that the Kaluza–Klein theories have to pay for such a geometrical picture of unification, consists of restrictions on the physically admissible spacetime an coordinates transformations. In fact, the spacetime has to take the structure

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of a generic 4-dimensional manifold plus a compact homogeneous hypersurface (such a feature implies a violation of the *Equivalence Principle*) and along the extra-dimensions only translations of the coordinates are available (so violating the *Principle of General Relativity* as extended to a multidimensional spacetime).

However, the geometrical theories of unification can be settled down within General Relativity by means of the so-called *Spontaneous Compactification* process (Witten, 1981). In this framework, the Lagrangian of the theory is yet the multidimensional Einstein–Hilbert one, but we observe the Kaluza–Klein restriction because the "vacuum state" has the compactified structure (i.e. it is a 4-dimensional Minkowski spacetimes a compact homogeneous manifold); thus, we may interpret the dimensional compactification as a spontaneous breaking of the Poincaré symmetry.

The aim of the present analysis is to extend the Kaluza–Klein approach even to the gauge connection associated to the Yang–Mills fields; it is achieved by splitting a free multidimensional spinor field Lagrangian and fixing the hypotheses necessary for the appearance (within the reduced 4-dimensional action) of the gauge connection terms.

While Sections 2 and 3 are devoted to review, respectively, the gauge theories and the Kaluza–Klein approach, in Section 4, as first step of this work, we generalize the Nöther theorem to the vacuum of a Kaluza–Klein theory; we show that the extra-dimensional components of the momentum operator correspond to the conserved charges associated to the gauge symmetry. Such an identification requires suitable hypotheses on the "matter" fields dependence with respect to the extra-coordinates; we take such a dependence in the form of a phase factor which include the gauge generators. In Section 5, we analyze the Dirac algebra on a Kaluza–Klein spacetime and see how the γ-matrices relations are preserved by the dimensional reduction process. The tangent space projection of the spinor connection is discussed to outline that its extra-dimensional component is a free quantity of our problem.

In Section 6, on the basis of a Lagrangian approach, we split the action of a free multidimensional spinor field and, after the dimensional reduction (here is crucial to take the integral over the extra-coordinates), we get the action of a 4-dimensional free spinor field plus the gauge coupling with the Yang–Mills fields; undesired terms appearing in the splitted action are removed by fixing the residual spinor connection components.

In Section 7 brief concluding remarks are provided to stress a difference existing between the abelian and non-abelian case.

2. GAUGE THEORIES

Let us assign, on the Minkowski space $\mathcal{M}^4$ (endowed with the coordinates system $\{x^\mu\}$ $\mu = 0, 1, 2, 3$), a set of fields $\phi_r(x^\mu)$ $(r = 1, 2, \ldots, n \in N)$, whose

Lagrangian density $\mathcal{L}(\phi_r,\ \partial_\mu\phi_r)$ is invariant under the unitary transformations

$$\phi_r = (\exp\{ig\omega^a T_a\})_{rs}\phi_s, \tag{1}$$

where ω^a $(a = 1, 2, \ldots, K \in N)$ denote constant parameters. while $T_{a\,rs}$ are the K-dimensional (Hermitian) group generators corresponding to the coupling constant g.

In agreement with the Nöther theorem (Mendl and Shaw, 1984; Coleman, 1988), this invariance implies the existence of the conserved charges

$$Q_a = ig\int_{\mathcal{E}^3} d^3x\pi_r T_{a\,rs}\phi_s, \tag{2}$$

π_r being the conjugate momentum to ϕ_r and $\mathcal{E}^3$ the three-dimensional Euclidean space. We upgrade the global symmetries (1), to a (local) gauge one, by requiring that their parameters become spacetime functions, i.e. $\omega^a = \omega^a(x^\mu)$. The invariance of the theory (i.e. of the Lagrangian) under the gauge symmetries involves new fields $A^a_\mu(x^\nu)$ into the dynamics (the so-called Yang–Mills fields); such fields, under infinitesimal gauge transformations (ω^a is replaced by $\delta\omega^a \ll 1$), behave as

$$A^a_\mu \to A^a_\mu - \varepsilon^{abc}\delta\omega^b A^c_\mu - \partial_\mu\delta\omega^a\,; \tag{3}$$

here the quantities ε^{abc} denote the structure constants of the Lie group and are defined via the relation $[T_a,\ T_b] = i\varepsilon^c_{ab}T_c$ (indices a, b, c are raised and lowered in Euclidean sense and repeated ones are summed from 1 to K). The fields A^a_μ are said abelian or non-abelian depending on whether the structure constants vanish or not.

In gauge invariant form, the Lagrangian density of the theory rewrites as

$$\mathcal{L}(\phi_r,\ D_\mu\phi_r) - \frac{g^2}{4}F^{a\,\mu\nu}F_{a\,\mu\nu}; \tag{4}$$

where $D_\mu\phi_r \equiv \partial_\mu\phi_r + igT_{a\,rs}A^a_\mu\phi_s$ indicates the *gauge covariant derivative* and the quadratic term in the gauge tensors $F^a_{\mu\nu} \equiv \partial_\nu A_\mu - \partial_\mu A_\nu + \varepsilon^{abc}A^b_\mu A^c_\nu$ provides the gauge vector fields dynamics.

3. KALUZA–KLEIN PARADIGM

Within a Kaluza–Klein theory (Appelquist *et al.*, 1987), the gauge fields are geometrized by adding, to the 4-dimensional spacetime $\mathcal{V}^4$ (having internal coordinates x^γ $\gamma = 0, 1, 2, 3$), a compact homogeneous D-dimensional space Σ^D (having a very small size and adapted coordinates θ^l $l = 4, \ldots, D$, whose isometries corresponds to the gauge symmetries. If we take the dimension of the gauge group equal to that one of the extra-space (i.e. $K = D$), then the whole manifold $\mathcal{V}^{4+D} = \mathcal{V}^4 4 \times \Sigma^D$ admits, in the $4 + D$-bein representation, the line element

$$ds^2 = \eta_{(C)(D)} e_A^{(C)} e_B^{(D)}\, dX^A\, dX^B \qquad \eta_{(C)(D)} = \mathrm{diag}\{1, -1, \ldots, -1\}, \tag{5}$$

being X^A, $A, B, = 0, 1, \ldots, D$ the coordinates on $\mathcal{V}^{4+D}$ (i.e. $X^A = \{x^\mu,\ \theta^l\}$), while the indices in parenthesis refer to the $4 + D$-bein; here the vectors $e_A^{(C)}$ take the form

$$e_\mu^{(A)} = \left(u_\mu^{(\nu)}(x^\gamma)\ ;\ \gamma A_\mu^{(n)}(x^\gamma)\right) \tag{6}$$

$$e_m^{(A)} = \left(\mathbf{0}; \xi_m^{(n)}(\theta^l)\right). \tag{7}$$

The reciprocal vectors $e_{(C)}^A$ (such that $e_A^{(C)} e_{(C)}^B = \delta_A^B;\ e_A^{(C)} e_{(D)}^A = \delta_{(D)}^{(C)}$) are as follows

$$e_{(\mu)}^A = \left(u_{(\mu)}^\nu(x^\gamma); -\gamma A_\nu^{(n)}(x^\gamma) u_{(\mu)}^\nu(x^\gamma) \xi_{(n)}^m(\theta^l)\right) \tag{8}$$

$$e_{(m)}^A = \left(\mathbf{0};\ \xi_{(m)}^n(\theta^l)\right)\ ; \tag{9}$$

here the vectors $u_{(\mu)}^\nu$ and $\xi_{(m)}^n$ are reciprocal respectively to $u_\mu^{(\nu)}$ and $\xi_m^{(n)}$, while $\gamma = \mathrm{constant}$.

The vectors $\xi_n^{(m)}$ correspond to the Killing fields of the compact manifold Σ^D and therefore satisfy the relations

$$\partial_n \xi_m^{(p)} - \partial_m \xi_n^{(p)} = C_{(q)(r)}^{(p)} \xi_m^{(q)} \xi_n^{(r)} \tag{10}$$

$${}^D\nabla_n \xi_m^{(p)} + {}^D\nabla_m \xi_n^{(p)} = 0\,, \tag{11}$$

where $C_{(q)(r)}^{(p)}$ denote the structure constants of the isometries group on Σ^D and the covariant derivative ${}^D\nabla_m$ refers to the extra-dimensional metric.

The Kaluza–Klein paradigm is implemented by requiring the explicit symmetry breaking of the $4+D$-dimensional diffeomorphisms into the 4-dimensional general coordinates transformations $x^{\mu'} = x^{\mu'}(x^\nu)$ and a translation along the extra-coordinates $\theta^{m'} = \theta^m + \omega^{(p)}(x^\nu)\xi_{(p)}^m$. Under infinitesimal transformations of the previous type, $u_\mu^{(\nu)}$ behave like 4-bein vectors, $\xi_m^{(n)}$ like scalar quantities and $A_\mu^{(m)}$ transform according to gauge 4-fields.

The geometrization of a gauge group is achieved by requiring that its structure constants ε^{abc} and the coupling constant g coincide, respectively, with those of the isometries $C^{(p)(q)(r)}$ and with the constant $\gamma\sqrt{c^3/16\pi\ {}^4G}$ (where the 4-dimensional Newton constant reads from the multi-dimensional one as ${}^4G \equiv {}^{4+D}G/V$, V being the volume of Σ^D). In fact, hence the $4 + D$-dimensional Einstein–Hilbert action provides, after dimensional reduction, the 4-dimensional Einstein-Yang–Mills one (i.e. we get ordinary 4-gravity and a Yang–Mills contribution).

4. NÖTHER THEOREM

We discuss the invariance of a 4+D-dimensional field theory, on the space-time $\mathcal{M}^4 \times \Sigma^D$ (in absence of gauge fields, i.e. $A_\mu^{(m)} \equiv 0$) under a translation of the coordinates, i.e. we extend the Nöther theorem (Mendl and Shaw, 1984) to the extra-dimensional context.

Let us consider a set of fields $\varphi_r(x^\mu, \theta^m)$ $(r = 1, 2, \ldots, n)$, whose Lagrangian density $\mathcal{L}$ is invariant under the infinitesimal coordinates displacement $x^{\mu'} = x^\mu + \delta\omega^\mu$ and $\theta^{m'} = \theta^m + \delta\omega^{(p)}\xi^m_{(p)}$ with $(\delta\omega^\mu\ \delta\omega^{(p)}) = \text{constant}$; in $4 + D$-dimensional notation, we take the infinitesimal coordinates transformation $X^{A'} = X^A + \delta\omega^{(B)}e^A_{(B)}$ (with $\delta\omega^{(A)} = (\delta\omega^\mu,\ \delta\omega^{(p)})$ and $u^\nu_{(\mu)} = \delta^\nu_{(\mu)}$) which, in turn, induces the corresponding fields transformation

$$\varphi'_r = \varphi_r + \delta\varphi_r, \qquad \delta\varphi_r = \partial_A\varphi_r e^A_{(B)}\delta\omega^{(B)}\,. \tag{12}$$

The Lagrangian density invariance provides

$$\partial_A \mathcal{L} e^A_{(B)}\delta\omega^{(B)} = \partial_A\left(\frac{\partial\mathcal{L}}{\partial(\partial_A\varphi_r)}\delta\varphi_r\right) - \left[\partial_A\left(\frac{\partial\mathcal{L}}{\partial(\partial_A\varphi_r)}\right) - \frac{\partial\mathcal{L}}{\partial\varphi_r}\right]\delta\varphi_r\,. \tag{13}$$

Now, using the Euler–Lagrange equations

$${}^{4+D}\nabla_A\left(\frac{\partial\mathcal{L}}{\partial(\partial_A\varphi_r)}\right) - \frac{\partial\mathcal{L}}{\partial\varphi_r} = 0 \tag{14}$$

and observing that, for $\mathcal{M}^4 \times \Sigma^D$, we have ${}^{4+D}\nabla_A e^A_{(B)} = 0$ (here ${}^{4+D}\nabla_A$ denotes the covariant derivative with respect to the metric J_{AB}), Eq. (12) rewrites as

$${}^{4+D}\nabla_A\left(\frac{\partial\mathcal{L}}{\partial(\partial_A\varphi_r)}\partial_C\varphi_r e^C_{(B)} - \mathcal{L}e^A_{(B)}\right) = 0\,. \tag{15}$$

This continuity equation leads to the conserved 4+D-dimensional fields momentum

$$P_{(A)} = \int_{\mathcal{E}^3\times\Sigma^D} d^3x d^D\theta\left\{\Pi_r\partial_B\varphi_r e^B_{(A)} - \mathcal{L}e^0_{(A)}\right\}, \tag{16}$$

where Π_r corresponds to the conjugate momenta of φ_r and behaves as $4 + D$-dimensional densities of weigh $1/2$.

The dependence of the fields φ_r on the extra-coordinates θ^m has to be in the form of a "phase" factor," because their matrix elements must not depend on them, i.e. we assume the structure

$$\varphi_r = \frac{1}{\sqrt{V}}e^{-i\tau_{rs}(\theta^m)}\phi_s(x^\mu)\ \Rightarrow \Pi_r = \frac{1}{\sqrt{V}}\pi_s(x^\mu)e^{i\tau_{sr}(\theta^m)}, \tag{17}$$

being ϕ_r and π_r 4-dimensional conjugate variables, while $\tau_{rs}(\theta^m)$ denoting generic Hermitian matrices compatible with the symmetries of Σ^D.

The substitution of these expressions into Eq. (16) provides the outcomings

$$Q_\mu \equiv P_\mu = \int_{\mathcal{E}^3} d^3x \left\{\pi_r \partial_\mu \phi_r - \mathcal{L}\delta^0_\mu\right\} \tag{18}$$

$$Q_{(m)} = -\frac{i}{V}\int_{\mathcal{E}^3\times\Sigma^D} \sqrt{\mathcal{K}} d^3x d^D\theta \left\{\pi_r \xi^n_{(m)} \partial_n \tau_{rs}(\theta^m)\phi_s\right\}, \tag{19}$$

where $\mathcal{K}$ refers to the determinant of the extra-dimensional metric.

We see that the 4-dimensional component of the conserved current corresponds to the ordinary 4-momentum vector. Furthermore, we take the position $\tau_{rs}(\theta^m) = T_{(p)\,rs}\lambda^{(p)}_{(q)}\Theta^{(q)}(\theta^m)$, $T_{(p)\,rs}$ being the gauge generators, $\lambda^{(p)}_{(q)}$ a constant matrix to be determined and $\Theta^{(q)}$ expandible in the harmonic functions of Σ^D (Salam and Strathdee, 1982); hence, the charges $Q_{(m)}$ rewrite

$$Q_{(m)} = -i\sqrt{\frac{c^3}{16\pi\ ^4G}}\int_{\mathcal{E}^3} d^3x\{\pi_r T_{(m)\,rs}\phi_s\}. \tag{20}$$

as soon as we identify the matrix $\lambda^{-1}{}^{(p)}_{(q)}$ with the quantities

$$\lambda^{-1}{}^{(p)}_{(q)} = \frac{\sqrt{16\pi\ ^4G}}{c^{3/2}V}\int_{\Sigma^D}\sqrt{\mathcal{K}} d^D\theta\left\{\xi^m_{(q)}\partial_m\Theta^{(q)}\right\}, \tag{21}$$

where by $\lambda^{-1}{}^{(p)}_{(q)}$ we denote the inverse matrix of $\lambda^{(p)}_{(q)}$ (i.e. $\lambda^{-1}{}^{(p)}_{(r)}\lambda^{(r)}_{(q)} = \delta^{(p)}_{(q)}$). Equation (20) coincides (apart from a factor $-\gamma$ which do not affect the conservation law) with the conserved quantities (2) and therefore it shows how, in a Kaluza–Klein theory, the charges associated to an abelian or non-abelian gauge theory come out from the extra-components of the fields momentum vector.

However, under the same assumption, Eqs. (12) and (17) provide, for the infinitesimal coordinates transformation $x^{\mu'} = x^\mu,\ \theta^{m'} = \theta^m + \delta\omega^{(p)}\xi^m_{(p)}$, the following gauge transformation on ϕ_r

$$\phi'_r = (\delta_{rs} - i\delta\omega^{(q)}\lambda^{(n)}_{(q)}\xi^m_{(n)}\partial_m\Theta^{(p)}T_{(p)\,rs})\phi_s. \tag{22}$$

Expression (22) becomes equivalent to (1) only if we require

$$\xi^m_{(n)}\partial_m\Theta^{(p)} = \delta^{(p)}_{(n)} \Rightarrow \xi^{(n)}_m = \partial_m\Theta^{(n)}. \tag{23}$$

This result is equivalent to the vanishing of all the structures constants $C^{(p)}_{(q)(r)}$. Thus, the extra-dimensional components of the 4+D-momentum vector become those 4-dimensional ones of a conserved charge in correspondence to abelian or non-abelian gauge theories; but the gauge transformation of the fields is induced by the translation along the extra-dimensions only for the abelian case, when the line element of Σ^D can be reduced to the Euclidean one, i.e. ${}^D dl^2 = \sum_{m=1}^D (d\Theta^m)^2$.

5. DIRAC ALGEBRA

In a flat 4+D-dimensional Minkowski space $\mathcal{M}^{4+D}$, the Lagrangian density of a set of massless spinor fields (the presence of a mass term does not affect the later analysis, while the chirality of the spinors is not addressed here (Wetterich, 1983)) $\Psi_r(X^A)$ takes the form

$$\mathcal{L}_\Psi = \frac{i}{2}(\partial_A \bar{\Psi}_r \gamma^A \Psi_r - \bar{\Psi}\gamma^A \partial_A \Psi_r), \tag{24}$$

where by γ^A ($\bar{\Psi} = \Psi^+\gamma^0$) we denote the Dirac matrices, satisfying the anti-commutation relations

$$\{\gamma_A,\ \gamma_B\} = 2I\eta_{AB}, \tag{25}$$

I being the identity matrix and η_{AB} the Minkowskian metric.

On a curved 4+D-dimensional spacetime, the Dirac matrices become functions on the manifold and have to be taken in the form $\gamma_A(X^B) = \gamma_{(B)}e_A^{(B)}$, being the 4+$D$-bein components equal to the constant matrices (24).

Thus, on a curved spacetime, the relation (25) rewrites as

$$\{\gamma_A(X^C)\ ,\ \gamma_B(X^C)\} = 2Ij_{AB}(X^C)\,, \tag{26}$$

In correspondence to the vectors (6) and (8) the matrices $\gamma^\mu = \gamma^{(\nu)}u^\mu_{(\nu)}$ and $\gamma_\mu = \gamma_{(\nu)}u_\mu^{(\nu)}$ define the appropriate 4-dimensional Dirac algebra with respect to the 4-metric $g_{\mu\nu} \equiv \eta_{(\rho)(\sigma)}u_\mu^{(\rho)}u_\nu^{(\sigma)}$. On a curved spacetime ${}^{4+D}\mathcal{V}$, the Lagrangian density (24) rewrites as

$$\mathcal{L}_\Psi^{\text{Curv}} = \frac{i}{2}\left(\mathcal{D}_A\bar{\Psi}_r\gamma^A\Psi_r - \bar{\Psi}\gamma^A\mathcal{D}_A\Psi_r\right) \quad \mathcal{D}_A \equiv \partial_A \pm \Gamma_A\,, \tag{27}$$

where $(-)$ and $(+)$ refer respectively to the application of the spinor derivative D_A on Ψ and $\bar{\Psi}$. The quantity Γ_A is a kind of "gauge connection" for the Lorentz group and reads

$$\Gamma_A = \Sigma^{(B)(C)}\Omega_{(B)(C)\,A} \tag{28}$$

$$\Omega_{(B)(C)\,A} \equiv e^D_{(C)}\ {}^{4+D}\nabla_A e_{(B)D} \tag{29}$$

$$\Sigma^{(A)(B)} \equiv \frac{1}{4}\left[\gamma^{(A)},\ \gamma^{(B)}\right]. \tag{30}$$

Here $\Sigma^{(A)(B)}$ is the generator of the Lorentz group in the spinor representation, while $\Omega_{(B)(C)\,A}$ plays the role of the corresponding six-gauge vectors (which in the Einstein theory can be expressed via the bein vectors $e_A^{(B)}$).

In agreement to the Spontaneous Compactification idea, within a Kaluza–Klein theory, the 4+D-dimensional Lorentz group is broken (near the "vacuum state" $\mathcal{M}^4 \times \Sigma^D$) into the 4-dimensional Lorentz one plus the D-dimensional translation group. Therefore, in this framework, the bein component of the quantity

(28) (i.e. $\Gamma_{(A)} \equiv \Gamma_B e^B_{(A)}$) has to admit only the 4-dimensional term of the form (28), i.e. $\Gamma_{(\mu)} = \Sigma^{(\nu)(\rho)} u^\alpha_{(\rho)} u^\beta_{(\mu)}\ {}^4\nabla_\beta u_{\alpha(\nu)}$.

The form of the bein component $\Gamma_{(m)}$ will be determined in the next section by requiring that, on $\mathcal{V}^4 \times \Sigma^D$, the Lagrangian density (27) provides the gauge coupling between the 4-spinors and the Yang–Mills fields.

6. GEOMETRIZATION OF THE GAUGE CONNECTION

We start from the following action for a set of $4+D$-dimensional spinor fields

$$S^{\text{Curv}}_\Psi = \frac{i}{2c} \int_{\mathcal{V}^4 \times \Sigma^D} d^4x d^D\theta \left\{ E(\mathcal{D}_{(A)} \bar{\Psi}_r \gamma^{(A)} \Psi_r - \bar{\Psi} \gamma^{(A)} \mathcal{D}_{(A)} \Psi_r) \right\}, \tag{31}$$

being $E \equiv \det e^{(B)}_A$. Recalling that $\gamma^{(A)} D_{(A)} \equiv \gamma^{(A)}(\partial_{(A)} \pm \Gamma_{(A)})$, we can split the previous action via the framework of Sections 4 and 5.

In fact, we have

$$\partial_{(A)} \equiv e^B_{(A)} \partial_B, \tag{32}$$

and hence, by (8), the following relations result

$$\partial_{(\rho)} = u^\mu_{(\rho)} \partial_\mu - \gamma u^\mu_{(\rho)} A^{(p)}_\mu \xi^m_{(p)} \partial_m \equiv {}^4\partial_{(\rho)} - \gamma u^\mu_{(\rho)} A^{(p)}_\mu \xi^m_{(p)} \partial_m \tag{33}$$

$$\partial_{(m)} = \xi^p_{(m)} \partial_p \equiv {}^D\partial_{(m)}; \tag{34}$$

where ${}^4\partial_{(a)}$ and ${}^D\partial_{(m)}$ denote the directional derivatives, respectively, on $\mathcal{V}^4$ and Σ^D.

Furthermore, we get

$$\gamma^{(A)} \Gamma_{(A)} = \gamma^{(\mu)} \Sigma^{(\rho)(\sigma)} \Omega_{(\rho)(\sigma)(\mu)} + \gamma^{(m)} \Gamma_{(m)} \tag{35}$$

$$\partial_{(m)} \Psi_r = -i \xi^n_{(m)} T_{(p)\,rs} \lambda^{(p)}_{(q)} \partial_n \Theta^{(q)} \Psi_s \tag{36}$$

$$\partial_{(m)} \bar{\Psi}_r = i \xi^n_{(m)} \bar{\Psi}_s T_{(p)\,sr} \lambda^{(p)}_{(q)} \partial_n \Theta^{(q)} \tag{37}$$

as well as $\bar{\Psi}_r \Psi_r = \bar{\psi}_r \psi_r / V$, being, in agreement with (17), $\{\bar{\psi}(x^\mu), \psi(x^\mu)\}$ the 4-dimensional spinor fields.

Putting together these results and taking in (31), the integral over the extra-coordinates, it provides the following 4-dimensional action

$$S^{\text{Curv}}_\Psi = \frac{1}{c} \int_{\mathcal{V}^4} d^4x U \left\{ \frac{i}{2} \left({}^4\mathcal{D}_{(\mu)} \bar{\psi}_r \gamma^{(\mu)} \psi_r - \bar{\psi} \gamma^{(\mu)}\, {}^4\mathcal{D}_{(\mu)} \psi_r \right) + \mathcal{L}_{\text{int}} \right\} \tag{38}$$

$$\mathcal{L}_{\text{int}} = g \bar{\psi}_r \gamma^{(\nu)} u^\mu_{(\nu)} A^{(m)}_\mu T_{(m)\,rs} \psi_s - \bar{\psi}_r \gamma^{(m)} \left(\sqrt{\frac{c^3}{16\pi\ {}^4G}} T_{(m)\,rs} - i V \Gamma_{(m)} \right) \psi_s, \tag{39}$$

where $U \equiv \det u_{\mu}^{(a)}$ and ${}^4\mathcal{D}_{(\mu)}$ denotes the ordinary 4-dimensional spinor derivative projected on the 4-bein.

In the previous equation, the first two terms provide the free 4-spinor fields, while those ones in S_{int} correspond respectively to the desired gauge coupling and to a contribution which is not experimentally detected; to remove such a term we need the choice $\Gamma_{(m)} = -i\frac{\sqrt{c^3/16\pi\ {}^4G}}{V} T_{(m)\,rs} I$.

At last, the previous equation rewrites

$$S_{\Psi}^{\text{Curv}} = \frac{1}{c}\int_{\mathcal{V}^4} d^4xU\left\{\frac{i}{2}\left(\left({}^4\mathcal{D}_{(\mu)} - igT_{(m)\,rs}A_{(\mu)}^{(m)}\right)\bar{\psi}_r\gamma^{(\mu)}\psi_s - \bar{\psi}_r\gamma^{(\mu)}\right.\right.$$
$$\left.\left.\times\left({}^4\mathcal{D}_{(\mu)} + igT_{(m)\,rs}A_{(\mu)}^{(m)}\right)\psi_s\right)\right\}. \tag{40}$$

We see how, after the dimensional reduction, the 4-dimensional action contains the correct gauge coupling, which appears as a consequence of the geometrical nature of the Yang–Mills fields within a Kaluza–Klein theory.

7. BRIEF CONCLUDING REMARKS

Putting together the geometrization of the Yang–Mills fields, performed in the usual Kaluza–Klein approach, with the result here obtained, we see that, starting from the $4{+}D$-dimensional gravity–matter action ${}^{4+D}S = {}^{4+D}S_{E-H} + {}^{4+D}S_{\Psi}^{\text{curv}}$, after the dimensional reduction, we get a 4-dimensional action describing all the appropriate bosonic and fermionic components with their relative couplings, i.e. (with obvious notation) ${}^4S = {}^4S_{E-H,\Lambda} + {}^4S_{Y-M}^{\text{curv}} + {}^4S_{\psi}^{\text{curv}} + {}^4S_{\text{int}}$.

The earlier analysis shows how, in the framework of a Kaluza–Klein theory, not only the Yang–Mills fields can be geometrized, but also their gauge couplings outcome from the splitting of the geometrical terms contained in the matter action.

The assumptions at the ground of our point of view are supported by the interpretation of the extra-dimensional components of the fields momentum in terms of gauge charges. However, a difference has to be emphasized between the abelian and non-abelian case; in fact, while for an abelian group the translations along the extra-dimensions induce directly the 4-dimensional gauge transformations, the latter one, in the non-abelian case can be recognized only in the structure of the (dimensionally reduced) 4-dimensional action. In this sense, the geometrization of the non-abelian gauge connection privileges the Lagrangian representation of the dimensional reduction with respect to an approach based on the field equations. Indeed, if we write down the $4{+}D$-dimensional Dirac equation and split it in terms of 4-dimensional variables, then to get the right gauge coupling terms it would be necessary to carry out an additional integration over the extra-dimensional coordinates; however, this same picture would appear when splitting the $4+D$-dimensional Einstein equations toward the 4-dimensional Einstein–Yang–Mills theory. Thus the different behavior, outlined earlier, between the abelian and non-abelian theories with respect to a geometrical interpretation, is a general feature of the Kaluza–Klein approach and it is not due to the specific assumptions we addressed here.

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